# Approximate Unsteady Thin-Airfoil Theory for Subsonic Flow

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**An analytical solution of the Possio integral equation is obtained for low reduced-frequency $\omega$, correct to order $\omega M/(1-M^2)$. The resulting loading, lift, and moment differ from that derived from GASP theory by Osborne, in agreement with recent work of Amiet, who has shown GASP theory to be inapplicable to two-dimensional flow with shed vorticity. The solution is applied to a generalized gust, a power law upwash, plunging motion, pitching motion, and a sinusoidal gust. Comparisons with numerical solutions are given for lift in the latter three cases. They show that one of the forms of the solution, the Osborne lift times a phase correction, is remarkably accurate up to $\omega M/(1-M^2)=\pi/4$. This approximation should prove convenient and useful in applications.**

## Introduction

THE nonsteady load distribution on two-dimensional thin airfoils oscillating in subsonic flow is governed by the Possio integral equation, which has no known exact analytical solution. There have been many numerical approaches to its solution over the years, and also a number of approximate analytical approaches. For a given airfoil motion, the forces depend on two parameters, the stream Mach number $M$ and the reduced frequency of oscillation $\omega$. The task of any approximate analytical solution is to give a satisfactory approximation in as large a region of the $M$, $\omega$ plane ($M<1$) as possible.

Some years ago, Miles[1] briefly discussed the compressibility correction to the incompressible problem (which has an exact solution). He transformed the governing differential equation and boundary conditions by a combined Galilean and Lorentz transformation, expanded to first order in $\omega M/\beta^2$, where $\beta^2=1-M^2$, and derived a compressibility correction rule. In another paper, Miles[2] derived a quasisteady theory by expanding the Possio integral equation to first order in $\omega$. Recently, Amiet and Sears[3] applied the method of matched asymptotic expansions to small-perturbation subsonic flows, and, again, obtained Miles' correction rule[1] in the course of their work, which has been called GASP (Glauert-Amiet-Sears-Prandtl) theory.

Osborne[4] applied this correction rule to obtain analytical formulas for the lift and moment of oscillating, thin two-dimensional airfoils flying in a generalized gust, a sinusoidal gust, and in plunging motion, although his results were in finite form only for the sinusoidal gust. Kemp[5] showed that the results for the generalized gust and the plunging motion also could be put in finite form, and added the other interesting case of pitching motion. (He also showed, subsequently[6], that the lift and moment for any integer power law upwash distribution could be derived by purely algebraic means from the results for the generalized gust.)

Amiet[7] examined the relation between Osborne's theory,[4] valid for small $\omega M/\beta^2$, and Miles' theory[2] valid for small $\omega$, and combined them to obtain an improvement to Osborne's theory. He applied the improved theory to the sinusoidal gust, and put the loading, lift, and moment in the form of Osborne's results times a phase factor which represented the improvement. When compared with exact calculations of lift at $M=0.6$, Amiet's formula showed considerable improvement over Osborne's result.

In a subsequent unpublished paper,[8] Amiet extended the comparison to the plunging airfoil and found his improvement of the Osborne-Kemp[5] lift to yeild good results for $M=0.8$ up to reduced frequencies of about 0.5. He also derived an approximate solution for large values of $\omega M/\beta^2$ which gave good results for lift caused by a gust.

The purpose of this paper is to extend the results of Refs. 7 and 8 in several directions. First, an approximate expression for the airfoil loading for low frequencies is derived directly from the Possio integral equation by expansion in powers of $\omega M/\beta^2$, correct to first order. The result is the same as Amiet's improvement of the Osborne theory, and shows directly that it is a consistent consequence of the Possio equation. Second, it is shown that this loading can be put into several forms, all of first order. One is Osborne's form with a redefinition of one of the parameters, which permits the lift and moment to be found easily by simple modifications of the formulas already derived by Kemp for Osborne's theory. Another form leads to a lift which is Osborne's modified by a phase factor. This generalizes to any upwash the sinusoidal gust result of Amiet.[7] The lift and moment in these various forms are presented for several upwash distributions.

Comparison of the lift caused by plunging, pitching, and a sinusoidal gust, with exact results shows the phase factor form to give very good results, as high as $M=0.8$.

The fact that the Osborne formulation is not a consistent low-frequency expansion of the exact two-dimensional result, but is missing a term, recently has been explained by Amiet.[13] He has shown that the GASP theory which Osborne used is inapplicable in the present case of two-dimensional flow with shed vorticity. For two-dimensional flow without shed vorticity, and three-dimensional flow, regardless of vorticity, the GASP theory is a consistent expansion.

## Approximate Integral Equation for Low Frequency

We take the oscillating motion to have the time dependence $e^{i\nu t}$, the half chord to be $b$ (the same as $c$ in Refs. 4-6), the stream density and velocity to be $\rho$ and $V$, and the upward

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loading on the airfoil to be $\Delta p$. The reduced frequency and modified reduced frequency are

$$\omega = \nu b/V \quad \omega^* = \omega/\beta^2 \quad \beta^2 = 1-M^2 \tag{1}$$

In these terms, a convenient form of the Possio integral equation is found in Eqs. (6-111) and (6-112) of Ref. 9. In terms of the upwash velocity $v$ to be cancelled by the loading $\Delta p$, these may be written, with $x$ measured in units of $b$ rearward from the airfoil center,

$$\beta v(x)\exp(-i\nu t - i\omega^* M^2 x) = \int_{-1}^{1} \beta^2 \frac{\Delta p(\xi)}{\rho V} \exp[-i\nu t - i\omega^* M^2 \xi] K^* d\xi \tag{2}$$

where

$$K^* = -\frac{\omega^*}{4}\left\{\frac{iM|x-\xi|}{(x-\xi)} H_1^{(2)}(\omega^* M|x-\xi|) - H_0^{(2)}(\omega^* M|x-\xi|) + i\beta^2 \exp[-i\omega^*(x-\xi)]\left[\frac{2}{\pi\beta}\ln\frac{1+\beta}{M} + \omega^*\int_0^{(x-\xi)} e^{i\omega^* u} H_0^{(2)}(\omega^* M|u|)du\right]\right\}$$

This equation has been written in a way which exhibits the variables in the form used in Ref. 3 to express the compressibility correction rule. In the associated "incompressible flow" $\beta v$ is the upwash, $\beta^2 \Delta p$ the loading, and $(\nu/\beta)T = \nu t + \omega^* M^2 f$ the time dependence. In those variables, the idea is to expand in powers of $\omega^* M$ and to keep terms of order $\omega^* M$ and $(\omega^* M)\ln(\omega^* M)$.

All the Hankel functions are seen to have arguments proportional to $\omega^* M$, and so they are expanded, using ($\gamma$ is Euler's constant)

$$H_0^{(2)}(z) = 1 - (2i/\pi)(\gamma + \ln z/2) + O(z^2)$$

$$H_1^{(2)}(z) = (2i/\pi z) - (iz/\pi)(\gamma - \tfrac{1}{2} + \ln z/2) + O(z^2)$$

where $O(z^2)$ may include $O(z^2 \ln z)$. The integral in $K^*$ is integrated by parts. Terms of order higher than $\omega^* M$ are dropped, and these include terms of the form $\omega^* M^2 O(\omega^*)$ which come from expanding the $\exp[-i\omega^*(x-\xi)]$ term and the integral. The result is

$$K^* = K^*_{\rm inc} - i\omega^* f(M)/2\pi + O(\omega^* M)^2 \tag{3}$$

where

$$K^*_{\rm inc} = \frac{1}{2\pi(x-\xi)} - \frac{i\omega^*}{2\pi}\exp[-i\omega^*(x-\xi)]\left[\frac{i\pi}{2} + \gamma + \ln\omega^*|x-\xi| + \int_0^{(x-\xi)} \frac{e^{i\omega^* u} - 1}{u} du\right] \tag{4}$$

$$f(M) = \beta\ln(1+\beta)/M + \ln M/2 = O(M^2) \tag{5}$$

The quantity $K^*_{\rm inc}$ is the exact incompressible ($M \to 0$) limit of $K^*$ except that $\omega$ is replaced by $\omega^*$. (If it is expanded for small $\omega^*$, then (3) agrees with Eqs. (3.1-3.3) of Ref. 2.) Thus, the approximate kernel (3) contains the exact incompressible kernel plus a compressibility correction, which is a constant during the integration.

The facts that the inversion of (2) is known for $K^* = K^*_{\rm inc}$, and that $K^*$ of (3) differs from $K^*_{\rm inc}$ by only a constant, permit us to invert (2) for the kernel of (3), as we will show next.

## Solution for Load Distribution

Suppose we know the integral transform pair (limits of all integrals are $-1$ to $+1$)

$$h(x) = \int g(\xi)K(x,\xi)d\xi \tag{6a}$$

$$g(x) = \int h(\xi)N(x,\xi)d\xi \tag{6b}$$

and we wish to solve

$$h'(x) = \int g'(\xi)[K(x,\xi) - A]d\xi \tag{7a}$$

where $A$ is a constant. If we write this as $h'(x) + A\int g'(\xi)d\xi = \int g'(\xi)K(x,\xi)d\xi$, then (6b) gives the inversion as

$$g'(x) = \int[h'(\xi) + A\int g'(y)dy]N(x,\xi)d\xi \tag{7b}$$

Though the function $g'$ being sought, appears on both sides, only its definite integral, a constant, is on the right. Thus, integration of (7b) permits us to solve for this integral as

$$\int g'(y)dy = \iint h'(\xi)N(y,\xi)d\xi dy[1 - A\iint N(y,\xi)d\xi dy]^{-1}$$

Then, putting this into (7b) gives the solution for $g'$ as

$$g'(x) = \int h'(\xi)N(x,\xi)d\xi + \frac{A\int N(x,\xi)d\xi \iint h'(\xi)N(y,\xi)d\xi dy}{1 - A\iint N(y,\xi)d\xi dy} \tag{8}$$

Using this formula, the integral equation (2) can be inverted for the approximate kernel (3).

However, in our case, (8) can be simplified, since $A = i\omega^* f(M)/2\pi$ is already small, $O(\omega^* M^2)$. Thus, the $A$ term in the denominator of (8) can be neglected, and it becomes

$$g'(x) = \int h'(\xi)[N(x,\xi) + A\int N(x,\eta)d\eta \int N(y,\xi)dy]d\xi \tag{9}$$

Further simplification also is possible, because, if $K = K^*_{\rm inc}$ in (6a), the $N$ of Eq. (6b) is the Schwarz inversion kernel for incompressible flow, with $\omega^*$ as the reduced frequency. But with $A = O(\omega^* M^2)$, only the part of $N$ independent of $\omega^*$ is needed in the $A$ term of Eq. (9). The rest of $N$ contributes terms of order $A\omega^* = (\omega^* M)^2$, which already have been neglected. Now, in the present notation the Schwarz kernel is [see p. 277, Eq. (5-342) of Ref. 9, example]

$$N(x,\xi) = \frac{-2}{\pi}\left[\frac{1-x}{1+x}\frac{1+\xi}{1-\xi}\right]^{1/2}\left[\frac{1}{x-\xi} + 1 - C(\omega^*)\right] - \frac{i\omega^*}{\pi}\ln\left\{\frac{1-\xi x - [(1-x^2)(1-\xi^2)]^{1/2}}{1-\xi x + [(1-x^2)(1-\xi^2)]^{1/2}}\right\} \tag{10a}$$

$$C(z) = K_1(iz)/[K_0(iz) + K_1(iz)] \tag{10b}$$

where $C$ is the Theodorsen function. The part of $N$ (say $N_0$) independent of $\omega^*$ is the first term, since $C - 1 = O(\omega^* \ln\omega^*)$. The integrals over this part are done easily, and lead to

$$A\int_{-1}^{1} N_0(x,\eta)d\eta \int_{-1}^{1} N_0(y,\xi)dy = 4A\left[\frac{1-x}{1+x}\;\frac{1+\xi}{1-\xi}\right]^{1/2} + O(\omega^* A) = -2\pi A[N(x,\xi)(x-\xi) + O(\omega^*)]$$

This permits (9) to be written as

$$g'(x)=\int_{-1}^{1} d\xi h'(\xi)\{N(x,\xi)[1-2\pi A(x-\xi)]+O(\omega^* A)\} \tag{11}$$

which is the final inversion formula for the integral equation (7a).

Comparison of (7a) and (2) allows the identification of $g'$ and $h'$, so (11) for our case gives the loading as

$$\frac{\Delta\bar{p}}{\rho V}=\int_{-1}^{1}\frac{\bar{v}}{\beta}\exp[i\omega^* M^2(x-\xi)]$$

$$N(x,\xi)[1-i\omega^* f(M)(x-\xi)]d\xi \tag{12}$$

where the overbar means the $e^{i\nu t}$ factor is omitted. An alternate form of the same order is obtained by writing

$$1-\omega^* f(M)(x-\xi)=\exp[-i\omega^* f(M)(x-\xi)]+O(\omega^* M^2)^2$$

which changes (12) to

$$\frac{\Delta\bar{p}}{\rho V}=\int_{-1}^{1}\frac{\bar{v}}{\beta}\exp[i\omega^*(M^2-f)(x-\xi)]N(x,\xi)d\xi \tag{13}$$

The loading formula (13) agrees with the one obtained by Amiet,[7] and (12) was implied in his note also. Now, (13) differs from the Osborne loading, obtained from GASP theory, by the presence of the $f$ terms, so Osborne's result appears to be inconsistent with an expansion to order $\omega^* M$, whereas Amiet's improvements is consistent to this order, and is the correct approximation.

The fact that (13) is in Osborne's form, except for replacement of $M^2$ by $M^2-f(M)$ in the exponent, permits all the formulas for lift and moment obtained from the Osborne theory[4-6] to be altered easily to incorporate this correction. These corrected formulas will be presented below, as well as those appropriate to the form (12).

First, let us find the load distribution which cancels a general upwash of the form

$$v(x,t)=v_0 e^{i\nu t}[A_0+2\sum_1^\infty A_n\cos n\Theta],\quad x=\cos\Theta \tag{14}$$

From (13) it is seen that the effective upwash is

$$w=ve^{-i\lambda^*\cos\Theta}\qquad \lambda^*=\omega^*[M^2-f(M)] \tag{15}$$

The well-known identify

$$e^{-iz\cos\Theta}=J_0(z)+2\sum_1^\infty(-i)^n J_n(z)\cos n\Theta \tag{16}$$

permits us to write $w$, using (14) and (16), as

$$w=v_0 e^{i\nu t}[A_0^*+2\sum_1^\infty A_n^*\cos n\Theta] \tag{17}$$

where the $A_n^*$ can be identified by series manipulation as[4]

$$A_0^*=A_0 J_0(\lambda^*)+2\sum_{m=1}^\infty(-i)^m A_m J_m(\lambda^*)$$

$$n>0:\ A_n^*=A_n J_0(\lambda^*)+\sum_{m=1}^\infty(A_{m-n}+A_{m+n})(-i)^m J_m(\lambda^*)$$

$$+\sum_{m=1}^n(A_{n-m}-A_{m-n})(-i)^m J_m(\lambda^*) \tag{18}$$

where $A_n=0$ for $n<0$. Integration of $wN$ yields $\Delta\bar{p}$ from Eq. (13) as

$$\frac{\Delta\bar{p}}{\rho V v_0}=\frac{\exp[i\lambda^*\cos\Theta]}{\beta}\left\{\frac{2(1-\cos\Theta)}{\sin\Theta}[A_0^*+(A_0^*+A_1^*)\right.$$

$$\left.(C(\omega^*)-1)]+\sum_1^\infty[4A_n^*+2i\omega^* n^{-1}(A_{n-1}^*-A_{n+1}^*)]\sin n\Theta\right\} \tag{19}$$

The pressure distribution thus may be found for any upwash which can be expressed as the Fourier series (14), by using (18) and (19). Equation (19) reduces to the incompressible loading to cancel the upwash (14), if $A_n^*$ is replaced by $A_n$, $\beta$ is put equal to 1, and $\lambda^*$ is put equal to 0.

The loading form (12) also can be used to express $\Delta p$ for the $v$ of (14) as a perturbation on Osborne's result. The first term of (12) was obtained by Osborne, and if we define

$$\lambda^0=\omega^* M^2 \tag{20}$$

then Osborne's loading $(\Delta p)_{Os}$ is the same as (19) with $\lambda^*$ replaced by $\lambda^0$. Since the $f$ term of (12) is already of order $\omega^* M^2$, the exponent multiplying it may be ignored, and $N$ may be replaced by $N_0$, the $\omega^*=0$ part of (10). Then the integration can be done using (14) for $v$, and the result is

$$\frac{\Delta\bar{p}}{\rho V}=\frac{(\Delta\bar{p})_{Os}}{\rho V}+\frac{2i\omega^* f}{\beta}v_0\left[\frac{1-x}{1+x}\right]^{1/2}(A_0+A_1) \tag{21}$$

With the expressions (19) and (21) for the pressure we can proceed to the lift and moment.

## Lift and Moment Expressions

The general expressions for lift and moment are found easily by integrating (19). It is convenient to express them in dimensionless terms as

$$\hat{L}=\frac{L(t)}{2\pi\beta^{-1}\rho b V v_0 e^{i\nu t}}\qquad \hat{M}=\frac{M(t)}{\pi\beta^{-1}\rho b^2 V v_0 e^{i\nu t}} \tag{22}$$

Then, from (19) we find the lift and nose-up moment about the mid-chord as

$$\hat{L}=\int_{-1}^{1}\frac{\Delta\bar{p}dx}{2\pi\rho V v_0\beta^{-1}}=(A_0^*+A_1^*)[C(\omega^*)(J_0(\lambda^*)-iJ_1(\lambda^*))$$

$$+iJ_1(\lambda^*)]+[(\omega^*/\lambda^*)-1]\sum_1^\infty i^n J_n(\lambda^*)(A_{n-1}^*-A_{n+1}^*) \tag{23a}$$

$$\hat{M}=-\int_{-1}^{1}\frac{\Delta\bar{p}xdx}{\pi\rho V v_0\beta^{-1}}=(A_0^*+A_1^*)[(J_0(\lambda^*)-J_2(\lambda^*))$$

$$(C(\omega^*-1)-2iJ_1(\lambda^*)C(\omega^*)]+2i\sum_1^\infty(A_{n-1}^*-A_{n+1}^*)i^n$$

$$\left[\left(\frac{\omega^*}{\lambda^*}-1\right)J_n'(\lambda^*)-\frac{\omega^*}{(\lambda^*)^2}J_n(\lambda^*)\right] \tag{23b}$$

These differ from Eqs. (15) and (20) of Ref. 5 in having $M^2$ replaced by $\lambda^*/\omega^*$, because the Osborne definition $\lambda^0=M^2\omega^*$ no longer holds. The present definition of $\lambda^*$ is given in (15), and includes $f(M)$.

Similar results are obtainable from (21). The first terms are just $\hat{L}_{Os}$ and $\hat{M}_{Os}$ obtained by putting $\lambda^0$ for $\lambda^*$ in (23a and

23b). For the lift, the second term integrates easily, so

$$\hat{L}=\hat{L}_{Os}+i\omega f(A_0+A_1) \qquad (24)$$

This can be put in a very convenient form by noting from Eq. (23a) and Eq. (18) that $\hat{L}_{Os}=(A_0+A_1)+O(\omega^*)$ so with an error of only $O(\omega^* M)^2$ we may write Eq. (24) as

$$\hat{L}=\hat{L}_{Os}(1+i\omega^* f)=\hat{L}_{Os}\exp(i\omega^* f) \qquad (25a)$$

This shows that the correction to Osborne's result can be expressed as a phase factor only, with the magnitude remaining unchanged. Amiet[7] used this form for the sinusoidal gust, and (25a) shows it to be true for any upwash of the form (14), and so for any upwash, since (14) is a general Fourier expansion.

For the moment, integration of (21) gives

$$\hat{M}=\hat{M}_{Os}+i\omega^* f(A_0+A_1) \qquad (26)$$

showing the correction to be one'half that of the lift. A form corresponding to (25a) is then

$$\hat{M}=\hat{M}_{Os}+\hat{L}_{Os}[\exp(i\omega^* f)-1] \qquad (25b)$$

Thus, the moment may differ from Osborne's result by more than just a phase factor, but it still is easy to calculate from the Osborne formulas.

The previously published special cases include the generalized gust,[5] the power-law upwash,[5] the plunging and pitching airfoil,[5] and the sinusoidal gust.[4,5] The calculations based on (19) leading to these results can be simply retraced retaining $\lambda^*$ rather than using Osborne's definition $\lambda^0$. The effect is to replace $\omega$ by $\omega^*-\lambda^*$ and $M^2$ by $\lambda^*/\omega^*$ in all the formulas except the sinusoidal gust, in which the Osborne definition of $\lambda^0$ led to simplifications not now applicable.

For the generalized gust whose upwash is

$$v(x,t)=v_0\exp[i\nu t-i\mu bx/V] \qquad (27)$$

we define

$$\lambda=\mu b/V \qquad \Lambda=\lambda+\lambda^* \qquad (28)$$

Then we have directly from (15, 27, and 16) that $A_n^*=(-i)^n J_n(\Lambda)$ in (17) so that (23) gives, using the summation formulas of Ref. 5,

$$\begin{aligned}\hat{L}_g&=[J_0(\Lambda)-iJ_1(\Lambda)]\\&\quad[C(\omega^*)(J_0(\lambda^*)-iJ_1(\lambda^*))+iJ_1(\lambda^*)]\\&\quad+i\lambda^{-1}(\omega^*-\lambda^*)[J_0(\lambda^*)J_1(\Lambda)-J_0(\Lambda)J_1(\lambda^*)] \qquad (29a)\end{aligned}$$

$$\begin{aligned}\hat{M}_g&=[J_0(\Lambda)-iJ_1(\Lambda)]\\&\quad[(C(\omega^*)-1)(J_0(\lambda^*)-J_2(\lambda^*))-2iJ_1(\lambda^*)C(\omega^*)]\\&\quad+\lambda^{-1}(\omega^*-\lambda^*)[2J_1(\Lambda)J_1(\lambda^*)+J_0(\Lambda)(J_0(\lambda^*)\\&\quad-J_2(\lambda^*))]+2\lambda^{-2}(\Lambda-\omega^*)[J_0(\lambda^*)J_1(\Lambda)\\&\quad-J_0(\Lambda)J_1(\lambda^*)] \qquad (29b)\end{aligned}$$

which agree with Eqs. (6) and (7) of Ref. 5 when $\lambda^*\to M^2\omega^*=\lambda^0$.

For an integer power law upwash of the form

$$v(x,t)=v_0e^{i\nu t}x^n \qquad (30)$$

the lift and moment can be obtained by expansion of (29) in powers of $\lambda$, as shown in Ref. 6. The results, in terms of $\lambda^*$ are

$$\begin{aligned}\hat{L}_n&=i^n(J_0^{(n)}-iJ_1^{(n)})[C(\omega^*)(J_0-iJ_1)+iJ_1]\\&\quad+i^{n+1}(n+1)^{-1}(\omega^*-\lambda^*)[J_0J_1^{(n+1)}-J_1J_0^{(n+1)}] \qquad (31a)\end{aligned}$$

$$\begin{aligned}\hat{M}_n&=i^n(J_0^{(n)}-iJ_1^{(n)})[C(\omega^*)(J_0-J_2-2iJ_1)\\&\quad-(J_0-J_2)]-i^{n+2}(n+1)^{-1}\Big\{J_0^{(n+1)}[(\omega^*-\lambda^*)\\&\quad(J_0-J_2)-2J_1]+2J_1^{(n+1)}[(\omega^*-\lambda^*)J_1+J_0]\Big\}+2i^{n+2}\\&\quad(\omega^*-\lambda^*)(n+2)^{-1}(n+1)^{-1}[J_0J_1^{(n+2)}-J_1J_0^{(n+2)}] \qquad (31b)\end{aligned}$$

where all the arguments of the $J$ Bessel functions are $\lambda^*$ and $J^{(n)}$ means the $n$-th derivative. These agree with Eqs. (10) and (11) of Ref. 6 for $\lambda^*=\lambda^0$.

One special case of interest is plunging motion, for which the upwash is a constant,

$$v(x,t)=v_0e^{i\nu t} \qquad (32)$$

This is the case $n=0$ of (30), and (31) gives

$$\begin{aligned}\hat{L}_0&=(J_0-iJ_1)[C(\omega^*)(J_0-iJ_1)+iJ_1]\\&\quad+\tfrac{1}{2}i(\omega^*-\lambda^*)[J_0(J_0-J_2)+2J_1^2] \qquad (33a)\end{aligned}$$

$$\begin{aligned}\hat{M}_0&=(J_0-iJ_1)C(\omega^*)(J_0-J_2-2iJ_1)+iJ_1(J_0-J_2)\\&\quad+(1-\omega^*/\lambda^*)J_0J_2+(1+\omega^*/\lambda^*)J_1^2 \qquad (33b)\end{aligned}$$

where, again, the $J$ Bessel functions have argument $\lambda^*$. These results are in agreement with Eqs. (12) and (13) of Ref. 5 when $\lambda^*=\lambda^0$.

For this case, it is possible to express not only the lift, but also the loading and moment, as simple perturbations of Osborne's results, since only $A_0$ of Eq. (14) does not vanish. Then the correction term in Eqs. (21) and (26) differ from the Osborne terms by higher-order quantities, so we have from them and (25a)

$$\Delta p_0=(\Delta p_0)_{Os}e^{i\omega^* f} \quad L_0=(L_0)_{Os}e^{i\omega^* f} \quad M=(M_0)_{Os}e^{i\omega^* f} \qquad (33c)$$

A second important special case is pitching motion, for which the upwash is linear, so that $n=1$ in (30), and (31) gives

$$v(x,t)=v_0e^{i\nu t}x \qquad (34)$$

$$\hat{L}_1=\tfrac{1}{2}\hat{M}_0 \qquad (35a)$$

$$\begin{aligned}\hat{M}&=\tfrac{1}{2}(J_0-J_2-2iJ_1)[C(\omega^*)(J_0-J_2-2iJ_1)-(J_0-J_2)]\\&\quad-i\Big\{\frac{\omega^*-\lambda^*}{3}J_0^2+\Big[\frac{4\lambda^*-\omega^*}{3(\lambda^*)^2}+\frac{2(\omega^*-\lambda^*)}{3}\Big]J_1^2\\&\quad-\Big(\frac{\omega^*}{(\lambda^*)^2}+\frac{\omega^*-\lambda^*}{3}\Big)J_0J_2\Big\} \qquad (35b)\end{aligned}$$

with $J_n=J_n(\lambda^*)$. Equation (35a) is a relation known to be true exactly in linearized subsonic flow,[12] and is preserved in this approximate theory. Equation (35b) corrects a misprint in Eq. (21) of Ref. 5, where the coefficient of the $J_0J_2$ term at the end of the equation should be

$$(M^2\lambda^i)^{-1}+(\omega/3), \text{[instead of } 1+(\omega/3)] \qquad (36)$$

Finally, the case of the sinusoidal gust

$$v(x,t)=v_0\exp[i\nu(t-bx/V)] \qquad (37)$$

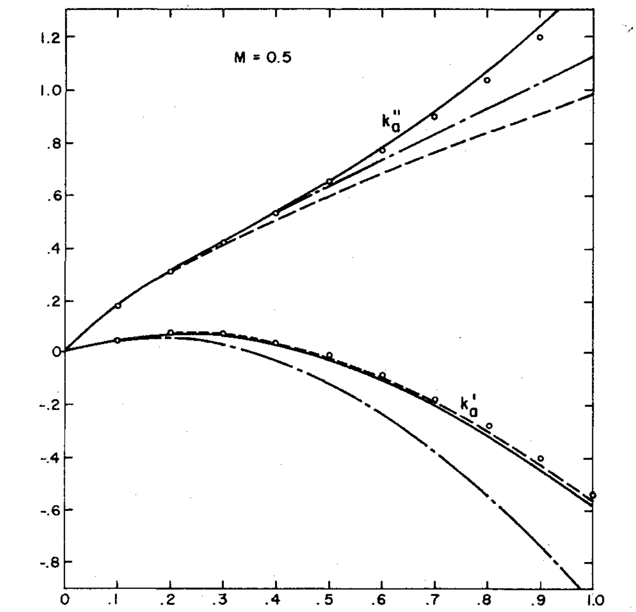


Fig. 1 Lift due to plunging motion at $M=0.5$:—— Numerical Results (Ref. 10),—•— Osborne, Eq. (33a)($\lambda^*=\lambda^0=\omega^*M^2$), – – – Corrected Osborne, Eq. (33a) [$\lambda^*=\omega^*(M^2-f)$] ∘∘∘∘ phase-corrected, Eqs. (33a) and (25a) ($\lambda=\lambda^0=\omega^*M^2$).

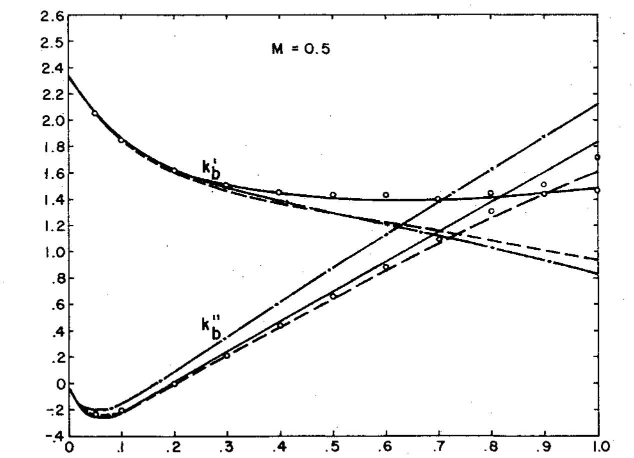


Fig. 2 Lift due to pitching motion about the quarter chord at $M=0.5$ (legend same as Fig. 1.)

is obtained easily from (29) by putting $\lambda=\omega$, so that (15, 28, and 1) show

$$\Lambda=\omega+\lambda^*=\omega^*[1-f(M)] \tag{38}$$

Equations (29) are to be used with this value of $\Lambda$. If $f(M)=0$, as in Osborne's theory, then Eqs. (29) simplify to

$$(\hat{L}_s)_{Os}=[J_0(\lambda^0)-iJ_1(\lambda^0)]S(\omega^*)$$

$$(\hat{M}_s)_{Os}=[J_0(\lambda^0)-J_2(\lambda^0)-2iJ_1(\lambda^0)]S(\omega^*)$$

$$S(z)=(iz)^{-1}[K_0(iz)+K_1(iz)]^{-1} \tag{39}$$

No such simplification is possible when $f\neq 0$, so the forms (25) with (39) are needed for the full calculation for that case. However, for this sinusoidal gust it is again possible to express not only the lift, but also the pressure and moment, as simple perturbations of Osborne's results, because $A_n=(-i)^n J_n(\omega)$, so all $A_n$ except $A_0$ are higher order in $\omega$. Therefore, again (21) and (26) may be written like (25a), to

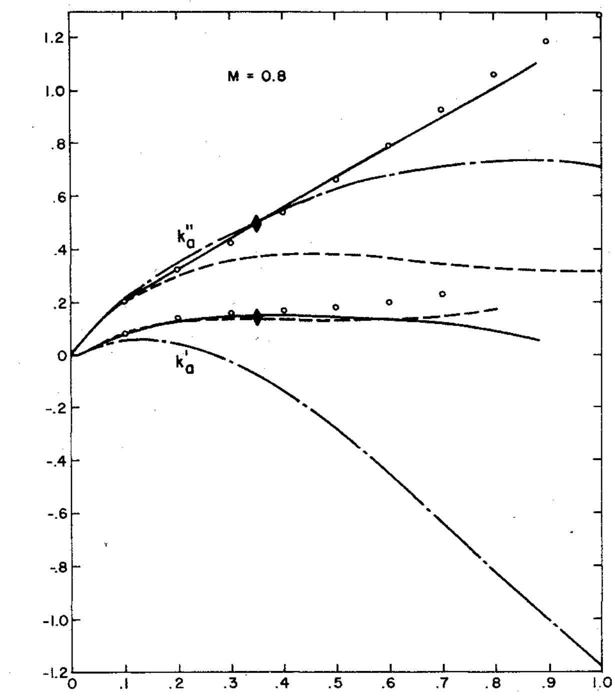


Fig. 3 Lift due to plunging motion at $M=0.8$. (Legend same as Fig. 1.)

give

$$\Delta p_s=(\Delta p_s)_{Os}e^{i\omega^*f}, \quad L_s=(L_s)_{Os}e^{i\omega^*f}$$

$$M_s=(M_s)_{Os}e^{i\omega^*f} \tag{40}$$

In the next section, lift caused by plunging, pitching, and the sinusoidal gust will be presented for several of these low-frequency approximations, and compared with the available exact numerical calculations.

## Results

In order to evaluate the accuracy of the various approximations, they will be compared with the exact numerical calculations of Refs. 10 and 11. In Ref. 10, a different notation is used, based on the displacement of the airfoil, rather than the upwash velocity. A lift coefficient for plunging displacement, $k_a=k_a'+ik_a''$, and a similar one for pitching displacement about the quarter chord $k_b$ are defined. The relation between them and the $\hat{L}_0$ and $\hat{L}_1$ used here can be shown to be

$$k_a'=-2\omega\beta^{-1}\mathrm{Im}(\hat{L}_0), \quad k_a''=2\omega\beta^{-1}\mathrm{Re}(\hat{L}_0)$$

$$k_a'=\beta^{-1}[2\mathrm{Re}(\hat{L}_0)-\omega\mathrm{Im}(\hat{L}_0)-2\omega\mathrm{Im}(\hat{L}_1)]$$

$$k_b''=\beta^{-1}[2\mathrm{Im}(\hat{L}_0)+\omega\mathrm{Re}(\hat{L}_0)+2\omega\mathrm{Re}(\hat{L}_1)]$$

We have calculated these coefficients for the Osborne approximation, using (33a, 33b, and 35a) with $\lambda^*\rightarrow\lambda^0$, for the corrected Osborne approximation using the same equations with $\lambda^*$, and for the phase-correction form using the Osborne formulas in (25a). These three approximations and the exact values are shown in Figs. 1-4 for $M=0.5$ and 0.8. Comparison shows that on an overall basis, the phase-correction form (25a) provides the best approximation to the exact values. In some cases, the corrected Osborne approximation is slightly better ($k_a'$), but, certainly, the phase-correction form is much better on the whole. A suggested range of accuracy, in $\omega$ and $M$, is for $\omega^*M<\pi/4$, which is $\omega<1.178$ for $M=0.5$ and $\omega<0.35$ at $M=0.8$. The latter point is marked on Figs. 3 and

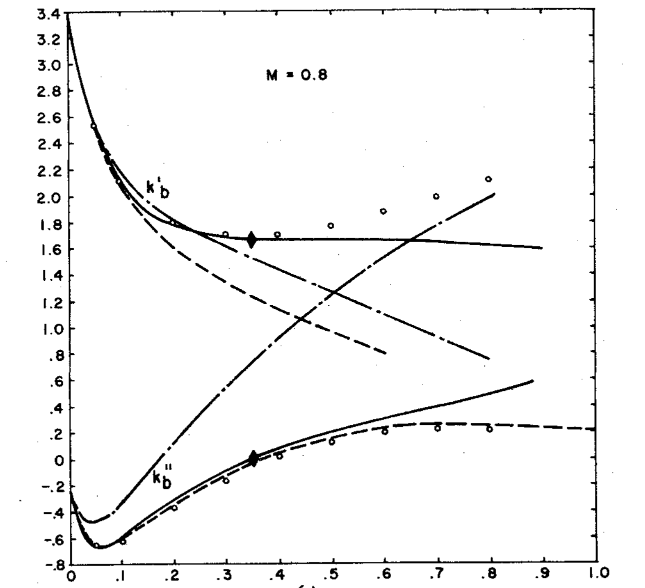


Fig. 4 Lift due to pitching motion about the quarter chord at $M = 0.8$. (Legend same as Fig. 1.)

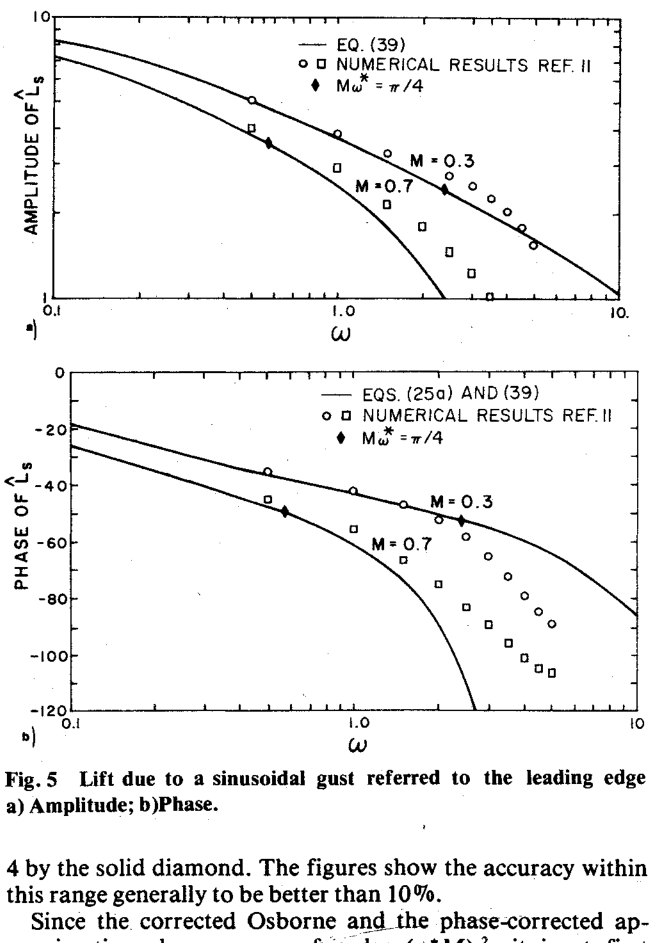


Fig. 5 Lift due to a sinusoidal gust referred to the leading edge a) Amplitude; b)Phase.

4 by the solid diamond. The figures show the accuracy within this range generally to be better than 10%.

Since the corrected Osborne and the phase-corrected approximations have errors of order $(\omega^*M)^2$, it is at first somewhat surprising that they deviate from each other considerably in some cases. In fact, however, for truly small values of $\omega^*M$, where $(\omega^*M)^2$ is really negligible, they are both nearly the same. For $(\omega^*M)^2 = 0.1$, $\omega = 0.21$ for $M = 0.5$ and $\omega = 0.14$ for $M = 0.8$. Within this range, all approximations are fairly good. The remarkable thing about the phase-correction form is that it holds up so well to $(\omega^*M)^2 = 0.6$ for all coefficients. At this point the accuracy of this approximation must be looked upon as somewhat fortuitous. Nevertheless, it certainly provides a useful and accurate engineering approximation for the lift. If moments are of interest, similar calculations should be made to check the accuracy of these various approximations to the moment.

The phase-correction approximation for the lift of a sinusoidal gust is shown as the line in Fig. 5, obtained from inserting (39) in (25a). (Actually, the line plotted is obtained by multiplying these formulas by $e^{-i\omega}$ so that the gust upwash is referred to the leading edge rather than the center of the airfoil.) Rather than the real and imaginary parts, the amplitude and phase are given. The points are exact numerical calculations from Ref. 11. Of course, the amplitude given in Fig. 5a is just the amplitude of the Osborne lift, Eq. (39), whereas the phase in Fig. 5b is the corrected phase according to Eq. (25a). Although the points, from the exact calculations of Ref. 11, are sparse at low frequency, it seems apparent that the phase-correction approximation is again a good one for frequencies below $\omega^*M = \pi/4$, marked by the solid diamonds. This is the same conclusion drawn by Amiet.[7]

## Conclusions

An expansion of the Possio intergral equation for low reduced frequency $\omega$(to order $\omega M/(1-M^2)$) enabled us to find an analytical solution to the same order for the loading on a thin subsonic airfoil subject to harmonic upwash of low frequency. The resulting lift and moment differ from that found by Osborne,[4] who applied GASP theory.[3] The difference is a term of order $\omega M^2/(1-M^2)$, which is the same as that found by Amiet[7] by a different method. The absence of this term in the Osborne application of the GASP theory has been explained by Amiet.[13] GASP theory is not applicable to two-dimensional flow with shed vorticity.

The present results were cast into two convenient forms, both the same to the order considered here. In one, the lift and moment are of the Osborne form,[4,5] but one of his parameters is modified to include the new term. In the second form, suggested originally for a sinusoidal gust by Amiet,[7] the lift differes from Osborne's only by a phase factor involving the new term, whereas the moment and loading are in general, that of Osborne plus a correction. However, for plunging motion as well as the sinusoidal gust, the moment and loading also differ from Osborne's by only phase factor.

The explicit formulas for lift and moment were given for a generalized gust, an upwash varying like any integral power of chordwise distance, constant upwash (plunging), linear upwash (pitching), and a sinusoidal gust. Comparisons of these approximate lift formulas with numerical calculations[10,11] were made for the plunging, pitching, and sinuosidal gust cases. It was found that the phase-corrected form provides remarkably accurate results (within 10%) for combinations of $\omega$ up to $\omega M((1-M^2) = \pi/4$. This form is a very good engineering approximation to lift in these cases, and should prove useful in applications.

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